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Markov-Type Inequalities for Polynomials with Restricted Zeros

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Upper bounds of the exact order of magnitude in n are given for

$$\frac{\max_{-1\leqslant x\leqslant 1}|p'(x)|}{\max_{-1\leqslant x\leqslant 1}|p(x)|}$$

for polynomials p of degree n, free of zeros in certain regions containing the interval $(-1, 1)$.

Key Words: polynomials; Markov inequalities; harmonic and subharmonic functions.

Markov's classical inequality

$$\|p'\|\leqslant n^2\,\|p\|,$$

where $\|\cdots\|$ stands for the maximum norm

$$\max_{-1\leqslant x\leqslant 1}|\cdots(x)|$$

over the interval $[-1, 1]$, is valid for all polynomials p of degree n with complex coefficients. The inequality is sharp for the nth Tchebyshev polynomial having all its zeros in $(-1, 1)$.

Erdős initiated in [1] the problem of improving the estimate under conditions ruling out this extremal case when he replaced the factor n^2 with cn for polynomials with zeros only on the two half-lines $(-\infty, -1]$ and $[1, \infty)$. Here and in what follows, c denotes absolute constants not necessarily the same at different occurrences. Apart from the value of c, this result is also best possible.

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Let D^α be the lens shaped region, the (open) circular biangle symmetric with respect to the real axis, bounded by two circular arcs joining 1 and -1 and meeting each other at an (inner) angle $\alpha\pi$ in ± 1. We prove the

THEOREM. *For every* $0 \leqslant \alpha < 1$ *there exists a constant* $c(\alpha)$ *depending only on* α *such that*

$$\|p'\| \leqslant c(\alpha)\, n^{2-\alpha}\, \|p\|$$

for polynomials p *of degree* n *non-vanishing in* D^α.[1]

Szegő's [2] general Markov-type inequality stated for D^α and the critical points ± 1,

$$|p'(\pm 1)| \leqslant c(\alpha)\, n^{2-\alpha} \sup_{z \in D^\alpha} |p(z)|,$$

has the same exponent in it. In fact, this formulation is a consequence of our Theorem if applied to

$$\sup_{z \in D^\alpha} |p(z)| + p(z)$$

that does not indeed vanish in D^α and has the same derivative as $p(z)$. Szegő proved his estimate best possible, the same then holds for $0 \leqslant \alpha < 1$ in our Theorem.

A similar construction shows that the symmetry of D^α plays no essential role, assuming in addition $p(z) \neq 0$ even for $\mathfrak{J} z < 0$ will result in no improvement in the Theorem.

The case $\alpha = 1$, i.e., that of the unit disc D^1, is, in fact, included if we apply the Theorem with $\alpha = 1 - 1/\log n$ and the fact from its proof that the constant factor can be chosen as

$$c(\alpha) = \frac{c}{1-\alpha}:$$

$$\|p'\| \leqslant cn \log n\, \|p\|$$

for polynomials non-vanishing in the unit disc.[2]

It is interesting to note that for polynomials with real coefficients $\log n$ is not necessary here: This is a special case of earlier results by Borwein and Erdélyi in [4], allowing among other things a given number of exceptional zeros inside the disc. See also the forthcoming paper [5] by Erdélyi

[1] I owe this problem to T. Erdélyi, who attributes it to P. Erdős.

[2] This observation for which I originally had a somewhat different proof has already appeared in [3].

exhibiting greater differences between real and complex cases than just a logarithmic factor. It would be interesting to carry over their investigations to D^α and even to more general regions.

If $1<\alpha<2$, then

$$\|p'\| \leqslant c(\alpha) n \, \|p\|$$

for polynomials with complex coefficients, as well. We shall give a hint of how to get it together with better pointwise bounds inside $(-1, 1)$ in the Theorem, at the end of this paper.

For other Markov-type inequalities see the recent books [3] and [6].

Proof. Without loss of generality we may assume

$$|p(x)| \leqslant 1 \qquad (-1 \leqslant x \leqslant 1)$$

and have to prove

$$|p'(x)| \leqslant c(\alpha) \, n^{2-\alpha} \qquad (-1 \leqslant x \leqslant 1).$$

We first estimate $p(y)$ on the rest of the real axis, i.e., for $|y|>1$.

Let us introduce the notation Ω^α for the complement of D^α with respect to the closed plane. If

$$p(z)=a \prod_{i=1}^{n} (z-z_i),$$

where $z_i \in \Omega^\alpha$, then

$$\log |p(y)| = \log |p(x)| - \log \left|\frac{p(x)}{p(y)}\right| \leqslant -\sum_{i=1}^{n} \log \left|\frac{x-z_i}{y-z_i}\right| \qquad (-1 \leqslant x \leqslant 1).$$

Integration with respect to a positive measure μ on $[-1, 1]$ normalized by $\mu([-1, 1])=1$ yields

$$\log |p(y)| \leqslant -\sum_{i=1}^{n} u(z_i, y) \leqslant -n \inf_{z \in \Omega^\alpha} u(z, y),$$

where

$$u(z, y) \stackrel{\text{def}}{=} \int_{-1}^{1} \log \left|\frac{z-x}{z-y}\right| d\mu(x) = \int_{-1}^{1} \log |z-x| \, d\mu(x) + \log \frac{1}{|z-y|}.$$

We want to choose a function $u(z, y)$, i.e., a measure μ with $\inf_{z \in \Omega^\alpha} u(z, y)$ as large as possible. Due to the special shape of $D^\alpha(\Omega^\alpha)$, this extremal problem is easily solved.

It follows from the general theory of subharmonic functions, especially from their Riesz representation (see, e.g., [7]), that $u(z, y)$, as a function of z, is subharmonic in the closed plane, harmonic outside $[-1, 1]$ with the exception of a logarithmic pole at $z = y$ (meaning, $u(z, y) - \log(1/|z - y|)$ is harmonic at $z = y$), $u(\infty, y) = 0$, and conversely, every such function has an integral representation in question with a normalized measure μ.

Let $G(z, y, \Omega^\alpha)$ be the Green function of Ω^α, i.e., the function vanishing on the boundary of Ω^α and harmonic inside Ω^α with the exception of a logarithmic pole at y as has been just described. We now show that

$$G(z, y, \Omega^\alpha) - G(\infty, y, \Omega^\alpha)$$

can be continued to a function $u(z, y)$. (It will then be clear that this is the optimal choice but, of course, we do not need this fact.) Since

$$\inf_{z \in \Omega^\alpha} G(z, y, \Omega^\alpha) = 0,$$

this will yield the inequality

$$\log |p(y)| \leqslant nG(\infty, y, \Omega^\alpha).$$

It is convenient to apply the linear transformation

$$w = \frac{z-1}{z+1}$$

carrying Ω^α into the angular region

$$\Omega_1^\alpha \stackrel{\text{def}}{=} \left\{ w : |\arg w| \leqslant \pi - \frac{\alpha\pi}{2} \right\},$$

$[-1, 1]$ into the negative axis $[-\infty, 0]$, ∞ into 1, y into

$$y_1 \stackrel{\text{def}}{=} \frac{y-1}{y+1},$$

and it suffices to check the corresponding properties of the Green function $G(w, y_1, \Omega_1^\alpha)$.

Mapping further Ω_1^α onto a half–plane by w^β,

$$\beta = \frac{1}{2-\alpha} \qquad (1/2 \leqslant \beta < 1),$$

this function is computed as

$$G(w,\, y_1,\, \Omega_1^\alpha) = \log\left|\frac{w^\beta + y_1^\beta}{w^\beta - y_1^\beta}\right|.$$

Here w^β, the regular branch that takes the value 1 at $w=1$, continues into the whole plane slit along the negative axis, giving also a continuation of $G(w, y_1, \Omega_1^\alpha)$ as a harmonic function there for $w \neq y_1$. It is in this step that we have made use of the assumption $\alpha < 1$, i.e., $\beta < 1$: w^β maps the slit plane in a one–to–one fashion onto an angular region avoiding the negative axis, thus producing no logarithmic pole other than y_1. $G(w, y_1, \Omega_1^\alpha)$ even extends continuously onto the negative axis $[-\infty, 0]$ and it remains to see that it is subharmonic there.

In a neighbourhood of every point of the negative axis other than the endpoints 0 and ∞, $G(w, y_1, \Omega_1^\alpha)$ can be continued harmonically from the upper half of the neighbourhood into the lower half and vice versa. $|w^\beta + y_1^\beta|$ decreases and $|w^\beta - y_1^\beta|$ increases as we do these continuations along a circle $|w| = R$. We see that the continued value is smaller than the actual value there, hence $G(w, y_1, \Omega_1^\alpha)$ can be thought of as the maximum of two harmonic functions and is, as such, subharmonic. The same follows for $w = 0$ and ∞ by continuity.

(We have appealed to subharmonic functions in order to avoid cumbersome computations. However, one may directly verify that the absolutely continuous measure given by

$$\frac{d\mu(x)}{dx} = \frac{2\beta \sin \beta\pi}{\pi(1-x^2)}\left(\frac{x_1}{y_1}\right)^\beta \left(\left|\left(\frac{x_1}{y_1}\right)^\beta e^{\beta\pi i} + 1\right|^{-2} + \left|\left(\frac{x_1}{y_1}\right)^\beta e^{\beta\pi i} - 1\right|^{-2}\right),$$

$$x_1 = \frac{1-x}{1+x} \qquad (-1 < x < 1)$$

does represent our $u(z, y)$.)

We conclude that

$$\log|p(y)| \leqslant nG(\infty,\, y,\, \Omega^\alpha) = nG(1,\, y_1,\, \Omega_1^\alpha) = n \log\frac{1+y_1^\beta}{1-y_1^\beta}.$$

For $1 < y \leqslant 2$

$$y_1^\beta = \left(\frac{y-1}{y+1}\right)^\beta \leqslant \frac{1}{\sqrt{3}},$$

$$\log\frac{1+y_1^\beta}{1-y_1^\beta} < \log(1+y_1^\beta) + \log(1+3y_1^\beta) < 4y_1^\beta < 4(y-1)^\beta.$$

For $y > 2$,

$$y_1^\beta = \left(\frac{y-1}{y+1}\right)^\beta = \left(1 - \frac{2}{y+1}\right)^\beta \leqslant 1 - \frac{1}{y+1},$$

$$\log \frac{1+y_1^\beta}{1-y_1^\beta} < \log 2 + \log(y+1) < 4(y-1)^\beta,$$

estimating crudely this time.

Together with similar bounds valid for $y < -1$ we thus have an estimate on the whole real line

$$\log |p(y)| \leqslant 4n(|y|-1)^\beta \qquad (|y| > 1),$$

while by assumption

$$\log |p(y)| \leqslant 0 \qquad (|y| \leqslant 1).$$

Fixing $-1 \leqslant x \leqslant 1$, where we want to estimate the derivative, it will be sufficient to use

$$\log |p(y)| \leqslant 4n\, |y-x|^\beta \qquad (-\infty < y < \infty).$$

The function

$$\Re \left(\frac{z-x}{i}\right)^\beta \qquad (\Im z \geqslant 0)$$

has boundary value

$$|y-x|^\beta \cos \frac{\beta\pi}{2} \qquad (-\infty < y < \infty)$$

on the real axis. Hence

$$h(z) \overset{\text{def}}{=} \frac{4n}{\cos \dfrac{\beta\pi}{2}}\, \Re \left(\frac{z-x}{i}\right)^\beta,$$

harmonic in the upper half-plane, majorizes $\log |p(z)|$ on the real line. This also holds at ∞,

$$\lim_{R \to \infty} \max_{\substack{|z| = R \\ \Im z \geqslant 0}} \{\log |p(z)| - h(z)\} = 0,$$

for the first term increases at most logarithmically, the second does at least as a power of R. By the maximum principle we then get

$$\log|p(z)| \leqslant h(z) \qquad (\Im z \geqslant 0).$$

This implies for $|z-x| \leqslant r$,

$$\log|p(z)| \leqslant \frac{4n}{\cos\frac{\beta\pi}{2}}\, r^{\beta},$$

$$|p(z)| \leqslant e^{(4n/\cos(\beta\pi/2))r^{\beta}}$$

first in the upper half-plane, but similarly in the lower one.

Cauchy's inequality for the derivative gives

$$|p'(x)| \leqslant \frac{\max_{|z-x|=r}|p(z)|}{r} \leqslant \frac{e^{(4n/\cos(\beta\pi/2))\,r^{\beta}}}{r},$$

and choosing

$$r \overset{\text{def}}{=} \left(\frac{\cos\frac{\beta\pi}{2}}{n}\right)^{1/\beta},$$

$$|p'(x)| \leqslant e^{4}\left(\frac{n}{\cos\frac{\beta\pi}{2}}\right)^{1/\beta} \overset{\text{def}}{=} c(\alpha)\, n^{2-\alpha}.$$

The proof is completed.

Using the full strength of our estimation for $\log|p(y)|$ and

$$h(z) \overset{\text{def}}{=} \frac{4n}{\sin\beta\pi}\, \Im((z+1)^{\beta}+(1-\bar{z})^{\beta})$$

as majorant, one gets the improved Bernstein–Markov-type inequality

$$|p'(x)| \leqslant c(\alpha)\min(n^{1/\beta}, n(1-x^{2})^{\beta-1}) \qquad (-1 \leqslant x \leqslant 1).$$

If $\alpha > 1$, then D^{α} is the union of two discs. Applying our method to both of them separately, it gives, in a natural way, bounds for $\log|p(y)|$ not on the real axis but rather on the two circular arcs, the symmetric images of

$(-1, 1)$ with respect to the periphery of the discs. (The resulting inequality is a transformed form of the elementary one

$$|p(z)| \leqslant |z|^n \, |p(z_1)| \qquad (|z| > 1),$$

where p is a polynomial of degree n having no zero inside the unit disc and $z_1 = 1/\bar{z}$ is the symmetric image of z with respect to the unit circle.) Using harmonic majorization in the three regions the two circular arcs and $[-1, 1]$ divide the plane into—as we did for the two half-planes in the above proof—one extends these bounds to

$$|p(z)| \leqslant e^{c(\alpha)r} \qquad (|z-x| \leqslant r, \; -1 \leqslant x \leqslant 1)$$

in the whole plane, implying

$$|p'(x)| \leqslant c(\alpha)n \qquad (-1 \leqslant x \leqslant 1),$$

as we remarked in our discussion.

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